Diffusion-controlled annihilation $A+B\rightarrow 0$: The growth of an A-particle island from a localized A source in the B-particle sea

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We present the growth dynamics of an island of particles A injected from a localized A source into a sea of particles B and dying in the course of diffusion-controlled annihilation $A+B\rightarrow 0$. We show that in the one-dimensional (1D) case the island grows unlimitedly at any source strength Λ, and the dynamics of its growth does not depend asymptotically on the diffusivity of B particles. In the 3D case the island grows only at $\Lambda>\Lambda_c$, achieving asymptotically a stationary state (static island). In the marginal 2D case the island unlimitedly grows at any Λ but at $\Lambda<\Lambda_*$ the time of its formation becomes exponentially large. For all cases the numbers of surviving and dying A particles are calculated, and the scaling of the reaction zone is derived.

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For the last two decades the reaction-diffusion system $A+B\rightarrow 0$, where unlike species A and B diffuse and irreversibly react in a d-dimensional medium, has acquired the status of one of the most popular objects of research. This attractively simple system, depending on the initial conditions, displays a rich variety of phenomena, and, depending on the interpretation of A and B (chemical reagents, quasiparticles, topological defects, etc.), it provides a model for a broad spectrum of problems [1]. A crucial feature of many such problems is the dynamical *reaction front*—a localized reaction zone which propagates between domains of unlike species. The simplest model of a reaction front, introduced more than a decade ago by Galfi and Racz [2], is a quasi-one-dimensional model for two initially separated reactants uniformly distributed on the left side $(x<0)$ and on the right side $(x>0)$ of the initial boundary. Taking the reaction rate in the mean-field form $R(x,t)=ka(x,t)b(x,t)$ (k being the reaction constant) Galfi and Racz discovered that in the long-time limit $kt\rightarrow\infty$ the reaction profile $R(x,t)$ acquires the universal scaling form

$$R = R_f \mathcal{Q}\left(\frac{x-x_f}{w}\right), \qquad (1)$$

where $x_f \propto t^{1/2}$ denotes the position of the reaction zone center, $R_f \propto t^{-\beta}$ is the height, and $w\propto t^{\alpha}$ is the width of the reaction zone. Subsequently, it has been shown [3–7] that the mean-field approximation can be adopted at $d \geqslant d_c=2$ [with logarithmic corrections in the two-dimensional (2D) case], whereas in 1D systems fluctuations play the dominant role. Nevertheless, the scaling law (1) takes place in all dimensions with $\alpha=1/6$ at $d\geqslant d_c$ and $\alpha=1/4$ at $d=1$, so that at any d the system demonstrates a remarkable property: on the diffusion length scale $L_D\propto t^{1/2}$ the width of the reaction front asymptotically contracts unlimitedly: $w/L_D\rightarrow 0$ as $t\rightarrow\infty$. Based on this property a general concept of the front dynamics, the quasistatic approximation (QSA), has been developed [3,5,8,9]. The QSA consists in the assumption that for sufficiently long times the dynamics of the front is governed by two characteristic time scales. One time scale $t_J=-(d\ln J/dt)^{-1}$ controls the rate of change in the diffusive current $J=J_A=J_B$ of particles arriving at the reaction zone. The second time scale $t_f\propto w^2/D$ is the equilibration time of the reaction front. Assuming $t_f/t_J \ll 1$ from the QSA in the mean-field case with equal species diffusivities $D_{A,B}=D$ it follows that [3,8]

$$R_f \sim J/w, \quad w\sim (D^2/Jk)^{1/3}, \qquad (2)$$

whereas in the 1D case w acquires the k-independent form $w\sim(D/J)^{1/2}$ [3,5,9]. The most important feature of the QSA is that w and R_f depend on t *only through the boundary current* $J(t)$, which can be calculated analytically without knowing the concrete form of $\mathcal{Q}$. On the basis of the QSA a general description of the system $A+B\rightarrow 0$ with initially separated reactants has been obtained for arbitrary diffusion coefficients [10]. These results are in full agreement with extended numerical calculations and have been generalized recently to the case of nonmonotonic front motion [11].

The purpose of the present paper is to apply the QSA to the long-standing problem of growth of an A-particle island from a localized A source in a uniform B-particle sea. This important problem was first analyzed by Larralde *et al.* [12] for the special case of a static sea ($D_B=0$). Assuming that diffusing A particles are injected at a single point into a reactive d-dimensional substrate B and instantaneously react with B upon contact, Larralde *et al.* have studied the growth dynamics of the reacted region radius $r_f(t)$ and the numbers of dying and surviving A particles. Considering the reaction front dynamics as a Stefan problem, they have, in particular, shown that at any source strength r_f asymptotically grows by the laws $(t\ln t)^{1/2}$, $t^{1/2}$, and $t^{1/3}$ at $d=1$, 2, and 3, respectively. Subsequently, those results were generalized to the cases of imperfect reaction [13] and diffusion with a bias [14]; however, as in [12], the B particles were always presumed “frozen.” In this Brief Report we present a theory of growth of a d-dimensional A island for the physically most important situation when both A and B particles are mobile. In the framework of the QSA we first consider the simplest “standard” case with equal species diffusivities, and then we extend the obtained results to the case of arbitrary nonzero

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diffusivities, thus revealing a rich general picture of the island growth for $d=1$, 2, 3.

Let particles A be injected at $t \geqslant 0$ with a rate Λ at the point $\vec{r}=\vec{0}$ of the uniform d-dimensional sea of particles B, distributed with a density ρ. Particles A and B diffuse with nonzero diffusion constants $D_{A,B}$ and upon contact annihilate with some nonzero probability, $A+B \to 0$. In the continuum version this process can be described by the reaction-diffusion equations

$$\frac{\partial a}{\partial t} = D_A \nabla^2 a - R + \Lambda \delta(\vec{r}), \quad \frac{\partial b}{\partial t} = D_B \nabla^2 b - R \tag{3}$$

with the initial conditions $a(r,0)=0$, $b(r,0)=\rho$, and the boundary condition $b(\infty,t)=\rho$. Here $a(r,t)$ and $b(r,t)$ are the mean local concentrations of A and B which, by symmetry, we assume to be dependent only on the radius, and $R(r,t)$ is the macroscopic reaction rate.

To simplify the problem essentially we will first assume, as usual, $D_A=D_B=D$. The initial density of the sea, ρ, defines a natural scale of concentrations and a characteristic length scale of the problem—the average interparticle distance $\ell = \rho^{-1/d}$. So, by measuring the length, time, and concentration in units of ℓ, ℓ^2/D, and ρ, respectively, we introduce the dimensionless source strength $\lambda = \Lambda \ell^2/D$ and the dimensionless reaction constant $\kappa = k/\ell^{(d-2)}D$. Defining then the difference concentration $s(r,t)=a(r,t)-b(r,t)$ we come from Eq. (3) to the simple diffusion equation with source

$$\partial s/\partial t = \nabla^2 s + \lambda \delta(\vec{r}) \tag{4}$$

at the initial and boundary conditions $s(r,0)=s(\infty,t)=-1$. According to Eq. (4) in the course of injection in the vicinity of the source there arises a region of A-particle excess, $s>0$, which expands with time. The central idea of the paper is that, by analogy with the Galfi-Racz problem, a narrow reaction front has to form at this region boundary, for which the law of motion, $r_f(t)$, according to the QSA, can be derived from the remarkably simple condition $s(r_f,t)=0$. Then, under the assumption that on the scale r_f the front width w can be neglected, $w/r_f \ll 1$, i.e., setting $a=s, b=0$ at $r<r_f$, whereas $a=0, b=|s|$ at $r>r_f$, the number of surviving $N_A(t)$ and that of dying $N_\times(t)$ A particles are immediately derived from the condition

$$N_A = \lambda t - N_\times = \Omega_d \int_0^{r_f} s(r,t) r^{d-1} dr \tag{5}$$

with $\Omega_1=2, \Omega_2=2\pi, \Omega_3=4\pi$. By calculating in the limit $w/r_f \ll 1$ the current of particles in the vicinity of the front, $J=-\partial s/\partial r|_{r=r_f}$, from Eq. (2) one can easily obtain the law $w(t)$ and define, in the end, a self-consistent condition of crossover to a quasistatic scaling regime (1). We start with an analysis of the behavior of $r_f(t)$, $N_A(t)$, and $N_\times(t)$ for each dimension separately.

In 1D the solution to Eq. (4) has the form

$$s(r,t) = (\sqrt{\lambda^2 t})\,\mathrm{ierfc}(r/2\sqrt{t}) - 1, \tag{6}$$

whence, according to the condition $s(r_f,t)=0$, the equation of motion of the reaction front center, $r_f(t)$, is

$$\mathrm{ierfc}(r_f/2\sqrt{t}) = 1/\sqrt{\lambda^2 t}, \tag{7}$$

where $\mathrm{ierfc}(\zeta)=\int_\zeta^\infty \mathrm{erfc}(v)dv = e^{-\zeta^2}/\sqrt{\pi} - \zeta\,\mathrm{erfc}(\zeta)$. From Eqs. (6) and (7) it formally follows that an excess of A particles forms in a time $t_c=\pi/\lambda^2$. It is, however, clear that a continual approximation comes into play at times $t \gg \max(1, 1/\lambda)$; therefore at early island formation stages one can distinguish two qualitatively different island growth regimes: (i) $\lambda \ll 1$, when the island formation proceeds under conditions of death of the majority of injected particles, and (ii) $\lambda \gg 1$, when the island forms long before the beginning of intensive annihilation. Let us consider first the limit $\lambda \ll 1$. In this limit the interval between injection acts, $\delta t_\lambda = 1/\lambda$, is quite large; therefore in the case of a perfect reaction each injected particle dies long before the next one appears until the distance to the nearest sea particle, $\sim \lambda t - \sqrt{t}$, becomes comparable with the characteristic diffusion length $\sqrt{\delta t_\lambda}$. From this, for the time of beginning of injected particle accumulation, we find $t_b \propto t_c \propto \lambda^{-2}$, which reveals the sense of t_c. Assuming $\epsilon=(t-t_c)/t_c \ll 1$ we have $\zeta_f = r_f/2\sqrt{t} \ll 1$, and taking κ not to be too small [9] ($\kappa > \sqrt{\lambda}$) from the fluctuation law $w \sim 1/\sqrt{J}$ and Eqs. (5)–(7) we obtain $r_f/w \sim \sqrt{N_A} \sim \epsilon/\sqrt{\lambda}$ and $t_f/t_J \propto \lambda^2$. Thus, the condition of crossover to the regime of a quasistatic front is $\epsilon \gg \sqrt{\lambda}$. Defining then a minimal island by the condition $r_f/w \sim N_A \sim 1$ for the island formation time, we find $t_b \sim t_c(1+\sqrt{\lambda})$ and conclude that at $\lambda \ll 1$ Eqs. (5)–(7) describe the island evolution over the whole interval from t_c to $t \to \infty$. In the long-time limit $\mathcal{T}=t/t_c \gg 1$ from Eq. (7) it follows that $\zeta_f \gg 1$ and we can rewrite Eq. (7) in the form $2e^{\zeta_f^2}\zeta_f^2 = \sqrt{\mathcal{T}}$, whence we obtain the *exact* asymptotics

$$r_f = \sqrt{2t \ln \mathcal{T}}(1 - \ln\ln \mathcal{T}/\ln \mathcal{T} + \cdots), \tag{8}$$

and from Eqs. (5), (6), and (8) we find $N_A = \lambda t[1 - O(\sqrt{\ln \mathcal{T}/\mathcal{T}})], N_\times = r_f(2+\zeta_f^{-2}+\cdots) = \sqrt{8t \ln \mathcal{T}}$. Consider now the limit $\lambda \gg 1$. It is evident that in this limit a multiparticle "cloud" forms long before the beginning of noticeable annihilation; therefore the stage of the developed reaction (8), $t \gg 1 (\gg t_c)$, is preceded here by a stage of purely diffusive expansion of the cloud, $1/\lambda \ll t \ll 1$. The statistical theory of diffusive cloud expansion has been developed recently in the work [15]. According to [15] in the "collective" regime ($\lambda t \gg 1$) the "radius" of a 1D cloud grows (in our units) by the law $r_+ = \sqrt{4t \ln(\lambda t)}$. Remarkably, r_+ and r_f "join" exactly in the front formation zone ($t \sim 1, r_+ \sim r_f$) whereas at $t \gg 1$ r_f begins to grow more slowly than r_+, as it has to. So, forming in qualitatively different regimes from $\phi = N_\times/N_A \gg 1$ ($\lambda \ll 1$) to $\phi \ll 1$ ($\lambda \gg 1$) the 1D island at any λ crosses over to the universal growth regime (8) with an unlimited decay of the dying particle fraction $\phi \propto \sqrt{\ln \mathcal{T}/\mathcal{T}} \to 0$. It remains for us to reveal the conditions of quasistaticity of the front (8). According to Eq. (6), in the limit $t, \mathcal{T} \gg 1$, the current $J \sim \sqrt{\ln \mathcal{T}/t}$. Thus assuming κ not to be too small ($\kappa > \sqrt{J}$) we find $w \sim J^{-1/2} \sim (t/\ln \mathcal{T})^{1/4}$, whence $w/r_f \sim (t \ln^3 \mathcal{T})^{-1/4}$ and $t_f/t_J \sim (t \ln \mathcal{T})^{-1/2}$. As $\zeta_f \gg 1$, the conditions $w/r_f, t_f/t_J \ll 1$ ought to be supplemented by a more strict requirement of equality of the currents at both sides of the front, $w \ll \mathcal{L} = -(d \ln J/dr)^{-1}|_{r=r_f} = r_f/2\zeta_f^2$. Calculating $w/\mathcal{L} \sim (\ln \mathcal{T}/t)^{1/4}$ we arrive at the requirement $t \gg \max(t_c, \ln \mathcal{T})$.

In 2D the solution to Eq. (4) has the form

$$s(r,t) = -(\lambda/4\pi)\mathrm{Ei}(-r^2/4t) - 1, \qquad (9)$$

whence, according to the condition $s(r_f,t)=0$, the equation of motion of the reaction front center $r_f(t)$ is

$$\mathrm{Ei}(-r_f^2/4t) = -4\pi/\lambda. \qquad (10)$$

Here $\mathrm{Ei}(-\zeta^2) = -\int_{\zeta^2}^{\infty} dv e^{-v}/v$ is the exponential integral, which has the asymptotics $\mathrm{Ei}(-\zeta^2) = \ln(\gamma\zeta^2) + \cdots$ at $\zeta^2 \ll 1$ ($\gamma = 1.781\ldots$) and $-e^{-\zeta^2}/\zeta^2 + \cdots$ at $\zeta^2 \gg 1$. From Eq. (10) it follows that r_f grows by the law

$$r_f = 2\sqrt{\alpha t}, \qquad (11)$$

where α is the root of the equation $\mathrm{Ei}(-\alpha) = -\lambda_*/\lambda, \lambda_* = 4\pi$ and has the asymptotics $\alpha = e^{-\lambda_*/\lambda}/\gamma$ at $\lambda \ll \lambda_*$ and $\alpha = \ln(\lambda/\lambda_*\alpha)$ at $\lambda \gg \lambda_*$. From Eqs. (5), (9), and (11) it follows that

$$N_A = \lambda t(1 - e^{-\alpha}), \quad N_\times = \lambda t e^{-\alpha}. \qquad (12)$$

We conclude that in 2D the island growth rate α and the dying-to-surviving particle ratio ϕ *do not vary in time*: at large $\lambda \gg \lambda_*$ the majority of particles survive, $\phi \sim \ln \lambda/\lambda$, whereas at small $\lambda \ll \lambda_*$, the majority of particles die, $\phi \sim e^{\lambda_*/\lambda}$. The most interesting consequence of Eq. (10) consists in the exponentially strong decrease of the growth rate in the region $\lambda < 1$. Defining a minimal island through the condition $N_A \sim 1$ for its formation time at $\lambda < 1$, we have $t_b \sim e^{\lambda_*/\lambda}/\lambda$, whence it is seen that at $\lambda \ll \lambda_*$ the island growth is actually suppressed. Calculating the current $J = \lambda/\lambda_* e^{\alpha}\sqrt{\alpha t}$ for the scaling of the reaction zone from Eq. (2) we find $w \sim (t/t_w)^{1/6}, t_w = (\kappa\lambda/e^{\alpha}\sqrt{\alpha})^2$. At $\lambda \ll \lambda_*$ this yields $\sqrt{t_f/t_J} \ll w/r_f \sim (t_b/\kappa t)^{1/3}$, whereas at $\lambda \gg \lambda_*$ we have $\sqrt{t_f/t_J} \ll w/\mathcal{L} \sim (\ln \lambda/\kappa t)^{1/3}$. Thus, crossover to the quasistatic regime occurs at times $\kappa t \gg t_b$ and $\kappa t \gg \ln \lambda$, respectively (note that $\kappa < 1$, being ~ 1 for a perfect reaction).

In 3D the solution to Eq. (4) has the form

$$s(r,t) = (\lambda/4\pi r)\mathrm{erfc}(r/2\sqrt{t}) - 1, \qquad (13)$$

whence, according to the condition $s(r_f,t)=0$, the equation of motion of the reaction front center $r_f(t)$ is

$$\mathrm{erfc}(r_f/2\sqrt{t}) = 4\pi r_f/\lambda. \qquad (14)$$

From Eq. (14) it follows that $\zeta_f(t)$ decreases indefinitely so that at large $t/t_s \gg 1$ the front radius, by the law $r_f = r_s[1 - O(\sqrt{t_s/t})]$ with the characteristic time $t_s = (\lambda/\lambda_*)^2$, reaches a stationary value

$$r_f(t/t_s \to \infty) = r_s = \lambda/\lambda_*. \qquad (15)$$

According to Eqs. (5), (13), and (14), in this limit $N_A = (2\pi/3)r_s^3[1 - O(\sqrt{t_s/t})], N_\times = \lambda t[1 - O(t_s/t)]$; therefore, in contrast to the 1D case, at any λ all the injected particles die. The steady-state current $J_s = \lambda_*/\lambda$, whence, according to Eq. (2), $w_s \sim (\lambda/\lambda_*\kappa)^{1/3}$ and $w_s/r_s \sim (\lambda_*/\lambda\sqrt{\kappa})^{2/3}$. Defining a minimal stationary island through the condition $w_s/r_s \sim 1$, we conclude that in the 3D case the island forms only when the injection rate exceeds a *critical* value $\lambda_c \sim \lambda_*/\sqrt{\kappa}$. The maximal value of κ, attainable in the perfect reaction limit, is $\kappa_p \sim \sigma/\ell$ (σ being the size of particles); therefore $\lambda_c > \lambda_* \gg 1$. By rewriting Eq. (14) in the form $\mathrm{erfc}(\zeta_f)/\zeta_f = 2\sqrt{t/t_s}$, one can easily see that at high injection rates $\lambda/\lambda_* \gg 1$ the stationary stage ($t \gg t_s$) is preceded by an intermediate stage $1 \ll t \ll t_s$ wherein the island grows by the law

$$r_f = \sqrt{2t \ln(t_s/t)}[1 - \ln(\sqrt{\pi}\omega)/\omega + \cdots], \qquad (16)$$

where $\omega = \ln(t_s/t)$. According to Eqs. (5), (13), and (16), at this stage $N_A = \lambda t[1 - O(\omega^{3/2}\sqrt{t/t_s})], N_\times = (4\pi/3)r_f^3 \times [1 + O(\omega^{-1})]$ and therefore the majority of particles are still surviving, $\phi \sim \omega^{3/2}\sqrt{t/t_s} \ll 1$. Calculating the current $J \sim \sqrt{\omega/t}$, we find from Eq. (2) $w \sim (t/\kappa^2\omega)^{1/6}$, whence $\sqrt{t_f/t_J} \ll w/\mathcal{L} \sim (\omega/\kappa t)^{1/3}$. Thus, the formed front condition reads $t \gg \ln(t_s/t)/\kappa$. According to [15] the radius of a 3D cloud, which expands in the absence of reaction, in our units has the form $r_+ = \sqrt{2t \ln[(\lambda\sigma/\ell)^2 t/4\pi]}$. Comparing r_+ and r_f suggests that r_f begins to lag behind r_+ at times $t \gg \ell/\sigma \sim 1/\kappa_p$, in remarkable agreement with the above estimation.

To sum up the above, as key results we distinguish (a) a *self-consistent analytic description* of the island growth from the moment of its formation and (b) *anomalously slow* island growth at $\lambda < \lambda_*$ in 2D and *complete suppression* of its growth at $\lambda < \lambda_c$ in 3D, which contrast sharply with the growth asymptotics in a static sea [12].

Let us now extend the analysis to the general case $0 < \mathcal{D} = D_B/D_A < \infty$, nondimensionality with respect to $D = D_A$ being retained. We present the final results here; a detailed discussion will be given elsewhere.

For $d=1$, comparing our results with the results of Ref. [12], we find that the long-t asymptotics (LTA) (8) for $\mathcal{D} = 1$ converges to the LTA for $\mathcal{D}=0$. We thus conclude that the growth of the 1D island does not depend asymptotically on the diffusivity of B particles. This conclusion is a consequence of the evident fact that at $r_f/\sqrt{\mathcal{D}t} \gg 1$ particles B can be regarded as effectively static, so as $r_f/\sqrt{t} \propto \sqrt{\ln t} \to \infty$ the LTA of the island growth at any $0 < \mathcal{D} < \infty$ must converge to the LTA for a static sea. In the interval $0 < \mathcal{D} < 1$ the time of crossover to the LTA does not alter appreciably, although the time of A-particle accumulation, t_b, shifts considerably at small λ (comparing $\lambda t - \sqrt{\mathcal{D}t}$ with $\sqrt{\delta t_\lambda}$ we find that in the interval $\mathcal{D} \ll \sqrt{\lambda} \ll 1$ the value of $t_b \propto \lambda^{-3/2}$ does not depend on $\mathcal{D}$, whereas at $\sqrt{\lambda} \ll \mathcal{D}$ it grows with $\mathcal{D}$ by the law $t_b \propto \mathcal{D}/\lambda^2$). At $\mathcal{D} \gg 1$ the time of crossover to the LTA becomes exponentially large, $t \gg e^{\mathcal{D}}/\lambda^2$; therefore the transient dynamics in this limit require special considerations.

In the 2D case the solution of the problem with a source possesses a remarkable property: $s(r,t) = f(r/\sqrt{t})$. Using this property and assuming $w/r_f(t \to \infty) \to 0$, it is easy to check that the solution of Eq. (3) has to read $a = -(\lambda/4\pi)\mathrm{Ei}(-\zeta^2) - \mathcal{A}, b=0$ at $r < r_f$ and $a=0, b = 1 + \mathcal{B}\mathrm{Ei}(-\zeta^2/\mathcal{D})$ at $r > r_f$, with $\zeta = (r/2\sqrt{t})(\mathcal{A}, \mathcal{B} = \mathrm{const})$. Equalizing the A and B currents at $r = r_f$, we come to the laws (11) and (12) with the *exact* equation for α at arbitrary $\mathcal{D}$:

$$\mathrm{Ei}(-\alpha/\mathcal{D}) = -(4\pi\mathcal{D}/\lambda)e^{\alpha(\mathcal{D}-1)/\mathcal{D}}. \qquad (17)$$

In the limit $\mathcal{D} \ll 1$ from Eq. (17) it follows that

$$\alpha \sim \begin{cases} \mathcal{D}e^{-\lambda_* \mathcal{D}/\lambda}/\gamma, & \lambda/\lambda_* \ll \mathcal{D}, \\ \lambda/\lambda_*, & \mathcal{D} \ll \lambda/\lambda_* \ll 1, \\ \ln(\lambda/\lambda_* \alpha), & \lambda/\lambda_* \gg 1. \end{cases}$$

In the opposite limit $\mathcal{D} \gg 1$ from Eq. (17) it follows that

$$\alpha \sim \begin{cases} \mathcal{D}e^{-\lambda_*/\tilde{\lambda}}/\gamma, & \tilde{\lambda}/\lambda_* \ll \ln^{-1}\mathcal{D}, \\ \ln[\tilde{\lambda}\ln(\mathcal{D}/\alpha)/\lambda_*], & \ln^{-1}\mathcal{D} \ll \tilde{\lambda}/\lambda_* \ll e^{\mathcal{D}}, \\ \ln(\lambda/\lambda_* \alpha), & \tilde{\lambda}/\lambda_* \gg e^{\mathcal{D}}, \end{cases}$$

where $\tilde{\lambda} = \lambda/\mathcal{D}$. Thus, for large differences of diffusivities, we find three characteristic growth regimes: $\alpha \ll \alpha_- = \min(1,\mathcal{D})$ (I), $\alpha_- \ll \alpha \ll \alpha_+$ (II), and $\alpha \gg \alpha_+ = \max(1,\mathcal{D})$ (III). In regime I, the majority of particles die, and the island growth rate does not depend on D_A. In regime III, the majority of particles survive, and the island growth rate does not depend on D_B. Leaving aside here the intermediate regime II, we distinguish the central fact: at any finite $\mathcal{D}$, as λ decreases, the island growth crosses over to the regime of "exponential suppression" I, which disappears only in the limit $\mathcal{D} \to 0$. According to Eq. (17) the boundary of this regime (with accuracy to a logarithmic factor at large $\mathcal{D}$) does not depend on D_A and in dimensional units is defined by the condition $\Lambda < \Lambda_* = 4\pi\rho D_B$.

For $d=3$, one can easily check that in the general case $\mathcal{D} \neq 0$, as in the case $\mathcal{D}=1$, the 3D island has to reach a stationary state asymptotically. Indeed, assuming $w_s/r_s \to 0$ in the steady-state limit $t \to \infty$ from Eq. (3), we find $a_s = \Theta(r_s - r)\mathcal{D}(r_s/r - 1), b_s = \Theta(r - r_s)(1 - r_s/r)$, where $\Theta(x)$ is the Heaviside step function and the stationary radius

$$r_s = \lambda/\lambda_* \mathcal{D} = \Lambda \ell^2/4\pi D_B$$

does not depend on D_A. For the particle number we find $N_A^s = (2\pi/3)\mathcal{D}r_s^3 \sim (\lambda/\lambda_*)^3/\mathcal{D}^2$, whence we obtain the lower island formation boundary $\lambda/\lambda_* > \mathcal{D}^{2/3}$ and conclude that the mean island density $\langle a \rangle_s = \mathcal{D}/2$ does not depend on λ: the island is always concentrated (with respect to the sea) at $\mathcal{D} \gg 1$ and is always rarefied at $\mathcal{D} \ll 1$. According to [10] for $\mathcal{D} \neq 1$ $w_s \sim (\mathcal{D}/\kappa J)^{1/3} \sim (r_s/\kappa)^{1/3}$, whence it follows that $w_s/r_s \sim (r_s^2\kappa)^{-1/3}$. Thus the necessary condition for 3D island formation takes the form

$$\lambda > \lambda_c \sim (\lambda_* \mathcal{D}/\sqrt{\kappa})\max(1, \sqrt{\kappa/\mathcal{D}^{2/3}}),$$

which in dimensional units reads $\Lambda > \Lambda_c \sim (4\pi D_B \sqrt{\rho D_A/k})\max[1, \sqrt{k(\rho/D_A D_B^2)^{1/3}}]$.

We have been unable to describe analytically the complete kinetics of crossover to the steady state for arbitrary $\mathcal{D} \neq 1$. However, in the interval $0 < \mathcal{D} < 1$ the intermediate asymptotics of 3D island growth can be revealed based on simple arguments. Indeed, at $\mathcal{D} < 1$ in the limit $\sqrt{\mathcal{D}t} \gg r_f$ we have $r_f \sim r_s$. In the opposite limit $\sqrt{\mathcal{D}t} \ll r_f$ the sea is effectively static; therefore the island ought to grow by the law $r_f \sim (\lambda t)^{1/3}$ [12]. From both conditions for the crossover time we find $t_s \propto \lambda^2/\mathcal{D}^3$ so that at $\mathcal{D} \ll 1$ in the interval $t < t_s$ the particle number grows by the law $N_A \propto (\lambda^{5/2} t)^{2/3}$ and the island density decays to $\langle a \rangle_s$ by the law $\langle a \rangle \propto (\lambda^2/t)^{1/3}$. In the limit $\mathcal{D} \to 0$ it follows that $\lambda_c \to 0, r_s \to \infty, t_s \to \infty$, and we are coming back to unlimited island growth at arbitrary finite λ.

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